

QUASICONFORMAL HARMONIC MAPPINGS BETWEEN $\mathcal{C}^{1,\mu}$ EUCLIDEAN SURFACES

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ABSTRACT. The conformal deformations are contained in two classes of mappings: quasiconformal and harmonic mappings. In this paper we consider the intersection of these classes. We show that, every K quasiconformal harmonic mapping between $\mathcal{C}^{1,\mu}$ ($0 < \mu \leq 1$) surfaces with $\mathcal{C}^{1,\mu}$ boundary is a Lipschitz mapping. This extends some recent results of several authors where the same problem has been considered for plane domains. As an application it is given an explicit Lipschitz constant of normalized isothermal coordinates of a disk-type minimal surface in terms of boundary curve only. It seems that this kind of estimates are new for conformal mappings of the unit disk onto a Jordan domain as well.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Let $u : D \rightarrow \mathbf{R}^n$ be a continuous mapping defined in a domain D of the complex plane $\mathbf{C} = \{z = x + iy, \ x, y \in \mathbf{R}\}$, given by

$$u(z) = (u^1(z), u^2(z), \dots, u^n(z)),$$

having partial derivatives in a point $z := x + iy \in D$. The formal derivative (Jacobian matrix) of u in z is defined by

$$\nabla u(z) = \begin{pmatrix} u_x^1 & u_y^1 \\ \vdots & \vdots \\ u_x^n & u_y^n \end{pmatrix}.$$

Then the Jacobian determinant (area magnification factor) of u and the Hilbert-Schmidt norm of $\nabla u(z)$ are defined by

$$J_u(z) = \left(\det[\nabla u(z)^T \cdot \nabla u(z)] \right)^{1/2} = \sqrt{|u_x|^2 |u_y|^2 - \langle u_x, u_y \rangle^2}$$

where by $\langle \cdot, \cdot \rangle$ and $|\cdot|$ are denoted the standard inner product and standard Euclidean norm in the space \mathbf{R}^n , and

$$\|\nabla u(z)\| = \left(\frac{1}{2} \text{Tr} \nabla u(z)^T \cdot \nabla u(z) \right)^{1/2} = \sqrt{\frac{1}{2}(|u_x|^2 + |u_y|^2)}.$$

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We also will use the following operator norm of ∇u :

$$|\nabla u(z)| = \max\{|\nabla u(z)h| : |h| = 1, h \in \mathbf{C}\},$$

and notice that, in the case of conformal mappings, the following two norms coincide.

1.1. Parametric Surfaces. We define an oriented parametric surface \mathcal{M} in \mathbf{R}^n to be an equivalence class of mappings $u = (u^1, \dots, u^n) : D \rightarrow \mathbf{R}^n$ of some domain $D \subset \mathbf{C}$ into \mathbf{R}^n , where the coordinate functions $u^k = u^k(x, y)$, $k = 1, \dots, n$ are of class at least $\mathcal{C}^1(D)$. Two such mappings $u : D \rightarrow \mathbf{R}^n$ and $\tilde{u} : \tilde{D} \rightarrow \mathbf{R}^n$, referred to as parametrizations of the surface, are said to be equivalent if there is a \mathcal{C}^1 -diffeomorphism $\phi : \tilde{D} \xrightarrow{\text{onto}} D$ of positive Jacobian determinant such that $\tilde{u} = u \circ \phi$. Let us call such ϕ a *change of variables*, or *reparametrization* of the surface. Furthermore, we assume that the branch (critical) points of \mathcal{M} are isolated. These are the points $(x, y) \in D$ at which the tangent vectors $u_x = \frac{\partial u}{\partial x}$, $u_y = \frac{\partial u}{\partial y}$ are linearly dependent. Equivalently, at the critical points the *Jacobian matrix* $\nabla u(z)$ has rank at most 1. It has full rank 2 at the *regular points*. A surface with no critical points is called an *immersion* or a *regular surface*. If $u : \mathbf{U} \rightarrow \mathcal{M}$, then the surface $\mathcal{M} = u(\overline{\mathbf{U}})$ is called a *disk-type* surface with boundary. If u has continuous extension to the boundary, then it defines the boundary $\partial\mathcal{M} := u(\mathbf{T})$ of the surface \mathcal{M} . If $u|_{\mathbf{T}} : \mathbf{T} \rightarrow \partial\mathcal{M}$ is a diffeomorphism, then the boundary is immersed.

The area of the surface equals

$$(1.1) \quad |u(D)| = \iint_D J_u \, dx \, dy.$$

1.2. Harmonic mappings. A mapping $u = (u^1, \dots, u^n) : D \rightarrow \mathbf{R}^n$ is called *harmonic* in a region $D \subset \mathbf{C}$ if for $k = 1, \dots, n$, u^k is real-valued harmonic functions in D ; that is u^k is twice differentiable and satisfies the Laplace equation

$$\Delta u^k := u_{xx}^k + u_{yy}^k = 0.$$

Let

$$P(r, t) = \frac{1 - r^2}{2\pi(1 - 2r \cos t + r^2)}, \quad 0 \leq r < 1, \quad 0 \leq t \leq 2\pi$$

denote the Poisson kernel. Then every bounded harmonic mapping $u : \mathbf{U} \rightarrow \mathbf{R}^n$, $n \geq 1$, defined on the unit disc $\mathbf{U} := \{z : |z| < 1\}$, has the following representation

$$(1.2) \quad u(z) = P[F](z) = \int_0^{2\pi} P(r, t - \varphi) F(e^{it}) dt,$$

where $z = re^{i\varphi}$ and F is a bounded integrable function defined on the unit circle $\mathbf{T} := \{z : |z| = 1\}$. If u is a harmonic mapping, then the surface $\mathcal{M} := u(\mathbf{U})$ is called a *harmonic surface*. If \mathcal{M} is a plane surface, then according to Lewy's theorem ([27]), $J_u > 0$ providing that u is a homeomorphism. On

the other hand, by a result of Berg ([3]) and Lichtenstein theorem, if \mathcal{M} is $C^{1,\alpha}$ regular, then $J_u > 0$ on D , providing that u is a homeomorphism.

1.3. Quasiconformal mappings. We say that a mapping $u : D \rightarrow \mathbf{R}^n$ is ACL (absolutely continuous on lines) in the region $D \subset \mathbf{C}$, if for every closed rectangle $R \subset D$ with sides parallel to the x and y -axes, u is absolutely continuous on a.e. horizontal and a.e. vertical line in R . Such a function has of course, partial derivatives u_x, u_y a.e. in D .

A homeomorphism $u : D \rightarrow u(D) \subset \mathbf{R}^n$ is said to be K -*quasiconformal* (K -q.c), $K \geq 1$, if u is ACL and if

$$(1.3) \quad \|\nabla u(z)\|^2 \leq \frac{1}{2} \left(K + \frac{1}{K} \right) J_u(z), \quad |z| < 1.$$

We will say that a q.c. mapping $f : \mathbf{U} \rightarrow \mathcal{M}$, of the unit disk onto a disk-type surface with rectifiable boundary $\gamma = \partial\mathcal{M}$ is *normalized* if $f(1) = w_0$, $f(e^{2\pi i/3}) = w_1$ and $f(e^{4\pi i/3}) = w_2$, where $\widehat{w_0 w_1}$, $\widehat{w_1 w_2}$ and $\widehat{w_2 w_0}$ are arcs of $\gamma = \partial\mathcal{M}$ having the same length $|\gamma|/3$, where $|\gamma|$ is the length of γ . (Such mappings have continuous extension to the boundary.)

Assume in addition that u is harmonic. Then

$$u(z) = (\operatorname{Re} a_1(z), \dots, \operatorname{Re} a_n(z)),$$

for some analytic functions $a_k(z)$, $z \in \mathbf{U}$, $k = 1, \dots, n$. Since

$$\|\nabla u(z)\|^2 = \frac{1}{2} \sum_{k=1}^n |a'_k(z)|^2,$$

it follows from (1.3) that $J_u(z) > 0$ except for some isolated points in \mathbf{U} which are branch points of the *harmonic quasiconformal surface* $\mathcal{M} := u(D)$.

1.4. Isothermal (conformal) parameters. In what follows, we will concern ourselves mostly conformal parametrizations (1 - quasiconformal mappings) $u = (u^1, \dots, u^n) : \Omega \rightarrow \mathbf{R}^n$. This simply means that the coordinate functions, called *isothermal parameters*, will satisfy the conformality relations: If $K = 1$ then (1.3) is equivalent to the system of the equations

$$\begin{cases} \sum_{k=1}^n u_x^k u_y^k = 0, & (u_x \text{ and } u_y \text{ are orthogonal in } \mathbf{R}^n) \\ \sum_{k=1}^n (u_x^k)^2 = \sum_{k=1}^n (u_y^k)^2 & (u_x \text{ and } u_y \text{ have equal length}). \end{cases}$$

Equivalently, it means that:

$$(1.4) \quad J_u = |u_x|^2 = |u_y|^2.$$

Thus u is an immersion if $\nabla u \neq 0$ at every point. We refer to [5] for an excellent historical account of existence of isothermal coordinates. If u is harmonic and satisfies the system (1.4), then \mathcal{M} is a minimal surface. A minimizing surface is a minimal surface spanning a given curve and having the least area.

Remark 1.1. A celebrated theorem by Lichtenstein states that each regular surface $\mathcal{M} := u(\Omega) \subset \mathbf{R}^n$ of class $\mathcal{C}^{1,\mu}$, $0 < \mu < 1$, can be mapped conformally onto a planar domain Ω . In particular, if \mathcal{M} is $\mathcal{C}^{1,\mu}$ disk-type surface with $\mathcal{C}^{1,\mu}$ immersed boundary, then there exists a conformal mapping τ of the unit disk onto \mathcal{M} such that

$$(1.5) \quad C_{\mathcal{M}} := \sup_{z \neq w} \frac{|\tau'(z) - \tau'(w)|}{|z - w|^\alpha} < \infty$$

and for some $c_{\mathcal{M}} > 0$

$$(1.6) \quad c_{\mathcal{M}} \leq |\tau'(z)| \leq \frac{1}{c_{\mathcal{M}}}, \quad z \in \mathbf{U},$$

and

$$(1.7) \quad \frac{1}{c_{\mathcal{M}}} \leq \frac{|\tau(z) - \tau(w)|}{|z - w|} \leq c_{\mathcal{M}}.$$

1.5. Some background and statement of the main result. The main question addressed here is under which conditions, a given harmonic diffeomorphism (or a given quasiconformal mapping) between smooth surfaces with smooth boundary is globally Lipschitz continuous. In the best known situation, where the domain and image domain is the unit disk, neither harmonic diffeomorphisms, neither quasiconformal mappings are Lipschitz. Under some additional conditions on the dilatation, a quasiconformal self-mapping of the unit disk is Lipschitz at the boundary (see [2]). Conformal mappings (minimal surfaces and minimizing surfaces) are a subclass of both classes: harmonic and quasiconformal mappings. Concerning the regularity and bi-Lipschitz character of minimal surfaces and minimizing surfaces we refer to [34], [11], [28] and [46]. In this paper, we will consider a bit more general situation. We will consider harmonic quasiconformal mappings between surfaces and investigate their Lipschitz character up to the boundary. This topic of research has its origin in the classical paper of Martio [32]. See also [7] and [6] for related results. In some recent results, see [21]-[25], [33], [31], [38]-[39] is established the Lipschitz and bi-Lipschitz character of quasiconformal harmonic mappings between smooth Jordan domains. In [43], [41] and [42] a similar problem for quasiconformal harmonic mappings with respect to the hyperbolic metric is treated. The class of quasiconformal harmonic mappings has been showed interesting, due to a recent discovery, that a q.c. harmonic mapping makes smaller distortion of moduli of annulus than a quasiconformal mapping [14].

Recently in [19], it is shown that, if u is a quasiconformal harmonic mapping of the unit disk onto a smooth $\mathcal{C}^{2,\mu}$ surface with $\mathcal{C}^{2,\mu}$ boundary, then u is Lipschitz.

In this paper we replace the condition $\mathcal{C}^{2,\mu}$ by $\mathcal{C}^{1,\mu}$ and find explicit Lipschitz constant for a normalized q.c. harmonic mapping. The celebrated Kellogg's theorem (see e.g. [8]), implies that a conformal mapping of the unit disk of onto a Jordan domain with $\mathcal{C}^{1,\alpha}$ boundary γ is Lipschitz continuous.

However, until now, it is not known any explicit estimation of Lipschitz constant, depending on γ .

Our main result can be stated as follows.

Theorem 1.2. *Let $u : \mathbf{U} \rightarrow u(\mathbf{U}) \subset \mathbf{R}^n$ be a K -quasiconformal harmonic mapping of the unit disk onto a $\mathcal{C}^{1,\mu}$ surface $\mathcal{M} = u(\mathbf{U})$ with $\mathcal{C}^{1,\mu}$ boundary γ . Then u is Lipschitz i.e. there exists a constant L such that*

$$|\nabla u(z)| \leq L, \quad z \in \mathbf{U}$$

and

$$|u(z) - u(w)| \leq KL|z - w|, \quad z, w \in \mathbf{U}.$$

If u is a normalized q.c. mapping, then L depends only on K and γ , and it not depends on u neither on the surface \mathcal{M} . It satisfies the inequality (3.17) below.

The proof of Theorem 1.2 is given in the third section. The proof rely on several lemmas which are proved in the second section, however the main ingredient of the proof is Lemma 3.1 (which can be considered as a Mori's theorem for q.c. mappings of the unit disk onto a smooth surface).

The proofs are different from the proofs of related results in [19] where is imposed $\mathcal{C}^{2,\mu}$ regularity of surface and some properties of the second derivative of conformal parametrization. Some of the tools for the proofs are conformal parametrization of a surface, arc-length parametrization of its boundary and isoperimetric inequality. We would like to point out Corollary 3.4 where is given an application of Theorem 1.2: it is given an explicit Lipschitz constant of isothermal coordinates of a minimal surface in terms of boundary curve only.

Recall that the family of quasiconformal harmonic mappings contains conformal mappings. In [29] is given an example of a \mathcal{C}^1 Jordan curve γ , such that a conformal mapping of the unit disk \mathbf{U} onto $D = \text{int}(\gamma) \subset \mathbf{C}$ is not Lipschitz. On the other hand, in [23] it is given an example of harmonic quasiconformal mapping of the unit disk onto itself, that is not smooth up to the boundary. This in turn implies that, our result is the best possible in this context.

2. AUXILIARY RESULTS

Lemma 2.1. *Let u be K -quasiconformal and $z = re^{it}$. Then*

$$(2.1) \quad \left| \frac{\partial u}{\partial t} \right|^2 \leq r^2 K J_u(z).$$

Proof. It is easily to obtain that, the condition (1.3) is equivalent to

$$(2.2) \quad |\nabla u(z)| \leq Kl(\nabla u(z)),$$

where

$$|\nabla u(z)| := \max\{|\nabla u(z)h| : |h| = 1\}$$

and

$$l(\nabla u(z)) := \min\{|\nabla u(z)h| : |h| = 1\},$$

(see e.g. [20, Lemma 2.1]). Let $h = (\alpha, \beta) \in \mathbf{T}$. Then

$$|\nabla u(z)h|^2 = |u_x|^2\alpha^2 + 2\langle u_x, u_y \rangle \alpha\beta + |u_y|^2\beta^2.$$

This yield

$$(2.3) \quad \max\{|\nabla u(z)h| : |h| = 1\} = \sqrt{\frac{(|u_x|^2 + |u_y|^2)(1 + \sqrt{1 - 4\eta^2})}{2}}$$

and

$$(2.4) \quad \min\{|\nabla u(z)h| : |h| = 1\} = \sqrt{\frac{(|u_x|^2 + |u_y|^2)(1 - \sqrt{1 - 4\eta^2})}{2}},$$

where

$$\eta = \frac{(|u_x|^2 \cdot |u_y|^2 - \langle u_x, u_y \rangle^2)^{1/2}}{|u_x|^2 + |u_y|^2}.$$

Therefore

$$(2.5) \quad J_u(z) = |\nabla u(z)| \cdot l(\nabla u(z)).$$

For $z = re^{it}$ we have

$$(2.6) \quad \frac{\partial u}{\partial t} = ru_y \cos t - ru_x \sin t.$$

Thus

$$(2.7) \quad rl(\nabla u) \leq \left| \frac{\partial u}{\partial t} \right| \leq r|\nabla u|.$$

By (2.7), (2.5) and (2.2) we obtain

$$(2.8) \quad \left| \frac{\partial u}{\partial t} \right|^2 \leq r^2 K J_u(z).$$

□

Definition 2.2. For a positive nondecreasing continuous function ω , $\omega(0) = 0$ we will say that is Dini's continuous if it satisfies the condition

$$(2.9) \quad \int_0^l \frac{\omega(t)}{t} dt < \infty.$$

A smooth Jordan curve γ , is said to be Dini's smooth if there exists a \mathcal{C}^1 diffeomorphism $h : \mathbf{T} \rightarrow \gamma$, such that the modulus of continuity ω of h' is Dini's continuous. Observe that every smooth $\mathcal{C}^{1,\mu}$ Jordan curve is Dini's smooth.

Lemma 2.3. *If ω is Dini continuous in $[0, l]$, then ω has Dini continuous extension in $[0, L]$, ($L > l$), which will be also denoted by ω . Moreover for every constant a , $\omega(ax)$ is Dini continuous. Next for every $0 < y \leq l$ there holds the following formula:*

$$(2.10) \quad \int_{0+}^y \frac{1}{x^2} \int_0^x \omega(at) dt dx = \int_{0+}^y \frac{\omega(ax)}{x} - \frac{\omega(ax)}{y} dx.$$

Proof. Taking the substitutions $u = \int_0^x \omega(at)dt$ and $dv = x^{-2}dx$, and using the fact that

$$\lim_{\alpha \rightarrow 0} \frac{\int_0^\alpha \omega(at)dt}{\alpha} = \omega(0) = 0$$

we obtain:

$$\begin{aligned} \int_{0+}^y \frac{1}{x^2} \int_0^x \omega(at)dt dx &= \lim_{\alpha \rightarrow 0+} \int_\alpha^y \frac{1}{x^2} \int_0^x \omega(at)dt dx \\ &= - \lim_{\alpha \rightarrow 0+} \left. \frac{\int_0^x \omega(at)dt}{x} \right|_\alpha^y + \lim_{\alpha \rightarrow 0+} \int_\alpha^y \frac{\omega(ax)}{x} dx \\ &= \int_{0+}^y \frac{\omega(ax)}{x} - \frac{\omega(ax)}{y} dx. \end{aligned}$$

□

Let $\gamma \in \mathcal{C}^1$, and $h : \mathbf{T} \rightarrow \gamma$ be a smooth function. We will sometimes write $h(t)$ and $h'(t)$ instead of $h(e^{it})$ and $\frac{d}{dt}h(e^{it})$. Consider the following function

$$(2.11) \quad \mathcal{K}_h(e^{is}, e^{it}) = \sqrt{|h(t) - h(s)|^2 |h'(s)|^2 - \langle h(t) - h(s), h'(s) \rangle^2}.$$

We need the following lemma.

Lemma 2.4. *If $h : \mathbf{T} \rightarrow \gamma$ is Dini's smooth function of the unit circle onto a Dini's smooth Jordan curve γ , and ω is modulus of continuity of h' , then*

$$(2.12) \quad |\mathcal{K}_h(e^{is}, e^{it})| \leq \frac{|h(e^{is}) - h(e^{it})|}{|e^{is} - e^{it}|} \int_0^{\pi|e^{is} - e^{it}|} \omega(\tau) d\tau.$$

In particular, if $h \in \mathcal{C}^{1,\mu}$, then

$$(2.13) \quad |\mathcal{K}_h(e^{is}, e^{it})| \leq c_h |h(e^{is}) - h(e^{it})| |e^{is} - e^{it}|^\mu,$$

where

$$c_h = \frac{1}{1 + \mu} \sup_{x \neq y} \frac{|h'(x) - h'(y)|}{|x - y|^\mu}.$$

Moreover, if $\varphi(e^{it}) = e^{if(t)}$ is a smooth mapping of \mathbf{T} onto itself, then

$$(2.14) \quad |\mathcal{K}_{h \circ \varphi}(e^{is}, e^{it})| = |f'(s)| |\mathcal{K}_h(e^{if(s)}, e^{if(t)})|.$$

Proof. Set $X = h(t) - h(s)$, $Y = h'(s)$ and $\alpha \in \mathbf{R}$. Then

$$\begin{aligned} &|X|^2 |Y + \alpha X|^2 - \langle X, Y + \alpha X \rangle^2 \\ &= |X|^2 (|Y|^2 + 2\alpha \langle X, Y \rangle + \alpha^2 |X|^2) - \langle X, Y \rangle^2 - 2\alpha \langle X, Y \rangle |X|^2 - \alpha^2 |X|^4 \\ &= |X|^2 |Y|^2 - \langle X, Y \rangle^2. \end{aligned}$$

Therefore we obtain

$$\begin{aligned}
\mathcal{K}_h(s, t) &= \sqrt{|X|^2|Y|^2 - \langle X, Y \rangle^2} \\
&= \sqrt{|X|^2|Y + \alpha X|^2 - \langle X, Y + \alpha X \rangle^2} \\
&\leq \sqrt{|X|^2|Y + \alpha X|^2} = |X||Y + \alpha X|.
\end{aligned}$$

Take now

$$\alpha = \frac{1}{s - t}.$$

Since

$$Y + \alpha X = h'(s) - \frac{h(t) - h(s)}{t - s} = \int_s^t \frac{h'(s) - h'(\tau)}{t - s} d\tau,$$

we obtain

$$\begin{aligned}
\left| h'(s) - \frac{h(t) - h(s)}{t - s} \right| &\leq \int_s^t \frac{|h'(s) - h'(\tau)|}{t - s} d\tau \\
&\leq \int_s^t \frac{\omega(\tau - s)}{t - s} d\tau \\
&= \frac{1}{t - s} \int_0^{t-s} \omega(\tau) d\tau.
\end{aligned}$$

As $|X| = |h(t) - h(s)|$, we obtain

$$(2.15) \quad |\mathcal{K}_h(e^{is}, e^{it})| \leq \frac{|h(s) - h(t)|}{|s - t|} \int_0^{|s-t|} \omega(\tau) d\tau.$$

Since $\mathcal{K}_h(e^{i(s \pm 2\pi)}, e^{i(t \pm 2\pi)}) = \mathcal{K}_h(e^{is}, e^{it})$, according to (2.15) and

$$\frac{1}{\pi} \min\{|s - t|, 2\pi - |s - t|\} \leq |e^{is} - e^{it}| \leq \min\{|s - t|, 2\pi - |s - t|\}$$

we obtain (2.12). The inequality (2.13) follows from (2.12). On the other hand

$$\frac{d}{dt} h(e^{if(t)}) = h'(f(t))f'(t)$$

and this yields (2.14). \square

Definition 2.5. Let $0 < \Upsilon \leq \pi$. A smooth surface \mathcal{M} is said to be Υ -isoperimetric if every rectifiable Jordan curve $\gamma \subset \mathcal{M}$ with the length L spans a subsurface $M_\gamma \subset \mathcal{M}$ with the area A satisfying the inequality

$$(2.16) \quad \frac{A}{L^2} \leq \frac{1}{4\Upsilon}.$$

2.1. Examples.

- Carleman, [4, Theorem 3.5, p. 129–232]. Every minimal surface (in particular every complex Jordan domain) is isoperimetric with $\Upsilon = \pi$.
- Courant, [4, Theorem 3.7. (The proof)]. Every harmonic surface is isoperimetric with $\Upsilon = 1$.
- Huber, [13]. If the integral mean of Gauss curvature K satisfies the inequality

$$\int_{\mathcal{M}} \max\{K, 0\} dM < 2\pi,$$

then

$$\Upsilon = \pi - \frac{1}{2} \int_{\mathcal{M}} \max\{K, 0\} dM.$$

- According to the previous item, every \mathcal{C}^2 surface is locally isoperimetric.
- Heinz, [10]. Every constant mean curvature surface $x = u(z)$, $z \in \overline{G}$ with mean curvature H satisfying $h := |H| \max_{z \in \overline{G}} |x - c| < 1$, $c \in \mathbb{R}^3$, is a isoperimetric surface with

$$\Upsilon = \pi \frac{1 - h}{1 + h}.$$

- Li&Tam [30]. Let M be a complete noncompact surface with finite total curvature. Suppose that all the ends of M have quadratic area growth. Then M is an isoperimetric surface for some constant Υ .
- Kalaj&Mateljević, [20]. Every quasiconformal harmonic surface with rectifiable boundary is a isoperimetric surface with $\Upsilon = \max\{\frac{2\pi}{1+K^2}, 1\}$.

Lemma 2.6. *If $u = P[F]$ is a harmonic mapping, such that F is a Lipschitz weak homeomorphism from the unit circle onto a Dini's smooth Jordan curve $\gamma = h(\mathbf{T})$ spanning a surface $u(\mathbf{U}) = \mathcal{M} \subset \mathbf{R}^n$, then for almost every $\tau \in [0, 2\pi]$ there exists*

$$J_u(e^{i\tau}) := \lim_{r \rightarrow 1} J_u(re^{i\tau})$$

and there holds the inequality

$$(2.17) \quad J_u(e^{i\tau}) \leq f'(\tau) \int_0^{2\pi} \frac{\mathcal{K}_h(e^{if(\tau)}, e^{if(t)})}{4\pi \sin^2 \frac{t-\tau}{2}} dt < \infty,$$

where

$$F(e^{it}) = h(e^{if(t)}),$$

and \mathcal{K}_h is defined in (2.11).

Proof. We will establish the existence of partial derivatives at the boundary of u_r and u_φ . Let

$$u(z) = (u_1(z), \dots, u_n(z)) = P[(F_1, \dots, F_n)].$$

Observe first that, for $i = 1, \dots, n$, there exists an analytic function h_i such that

$$u_i(z) = h_i(z) + \overline{h_i(z)} = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it}}{e^{it} - z} F_i(e^{it}) dt + \frac{1}{2\pi} \int_0^{2\pi} \frac{z}{e^{it} - z} \overline{F_i(e^{it})} dt.$$

It follows that

$$\begin{aligned} (2.18) \quad zh'_i(z) &= \frac{z}{2\pi} \int_0^{2\pi} \frac{e^{it}}{(e^{it} - z)^2} F_i(e^{it}) dt = \frac{1}{2\pi i} \int_0^{2\pi} \frac{z}{e^{it} - z} F'_i(e^{it}) dt \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{e^{it}}{e^{it} - z} F'_i(e^{it}) dt. \end{aligned}$$

Now in view of the fact that $\operatorname{Re} \frac{e^{i\tau}}{e^{i\tau} - z} > 0$, according to ([6, Theorem 2.2], see also [26]), we get that there exist radial boundary values of the function $zh'_i(z)$ almost everywhere.

It follows that, there exists

$$J_u(e^{i\tau}) := \lim_{r \rightarrow 1} J_u(re^{i\tau})$$

for almost every $\tau \in [0, 2\pi]$.

Now by using the fact that $F' \in L^1(\mathbf{T})$ we get

$$(2.19) \quad \lim_{r \rightarrow 1} u_\tau(re^{i\tau}) = u_\tau(e^{i\tau})$$

for almost every $e^{i\tau} \in \mathbf{T}$.

Let $e_3(r, \varphi), e_4(r, \varphi), \dots, e_n(r, \varphi)$ be an orthonormal system of vectors orthogonal to u_r and u_φ in the Euclidean space \mathbf{R}^n .

Define the vector

$$v(r, \varphi) = u_r(re^{i\varphi}) \times u_\varphi(re^{i\varphi}) \times e_3(r, \varphi) \times \dots \times e_{n-1}(r, \varphi).$$

Then $v(r, \varphi)$ is collinear with $e_n(r, \varphi)$ and its norm is given by

$$|v(r, \varphi)| = J_u(re^{i\varphi}).$$

Now for constant vectors $\chi_2, \dots, \chi_{n-2}$ and for the vector functions $\chi(t)$ and $\psi(t)$ there hold

$$(2.20) \quad \int_a^b \chi(t) dt \times \chi_2 \times \dots \times \chi_{n-2} = \int_a^b \chi(t) \times \chi_2 \times \dots \times \chi_{n-2} dt,$$

and

$$(2.21) \quad \left| \int_a^b \psi(t) dt \right| \leq \int_a^b |\psi(t)| dt.$$

By using (2.20), (2.21) and the fact that u is a harmonic mapping we obtain:

(2.22)

$$\begin{aligned}
\lim_{r \rightarrow 1} J_u(re^{i\tau}) &= \lim_{r \rightarrow 1} \left| u_r(re^{i\tau}) \times u_\tau(re^{i\tau}) \times e_3(r, \tau) \times \cdots \times e_{n-1}(r, \tau) \right| \\
&= \lim_{r \rightarrow 1} \left| \frac{u(re^{i\tau}) - u(e^{i\tau})}{1-r} \times u_\tau(e^{i\tau}) \times e_3(r, \tau) \times \cdots \times e_{n-1}(r, \tau) \right| \\
&= \lim_{r \rightarrow 1} \left| \int_{-\pi}^{\pi} (u(e^{it}) - u(e^{i\tau})) \times u_\tau(e^{i\tau}) \times \cdots \times e_{n-1}(r, \tau) \frac{P(r, \tau - t)}{1-r} dt \right| \\
&\leq \lim_{r \rightarrow 1} \int_{-\pi}^{\pi} \mathcal{K}_F(e^{i(t+\tau)}, e^{i\tau}) \frac{P(r, t)}{1-r} dt,
\end{aligned}$$

where $P(r, t)$ is the Poisson kernel and

$$\begin{aligned}
\mathcal{K}_F(e^{it}, e^{i\tau}) &= \sqrt{|u(e^{it}) - u(e^{i\tau})|^2 |u_\tau(e^{i\tau})|^2 - \langle u(e^{it}) - u(e^{i\tau}), u_\tau(e^{i\tau}) \rangle^2} \\
&= \sqrt{|F(e^{it}) - F(e^{i\tau})|^2 |F'(e^{i\tau})|^2 - \langle F(e^{it}) - F(e^{i\tau}), F'(e^{i\tau}) \rangle^2} \\
&= f'(\tau) \sqrt{|h(f(t)) - h(f(\tau))|^2 - \langle h(f(t)) - h(f(\tau)), h'(f(\tau)) \rangle^2},
\end{aligned}$$

i.e.

$$(2.23) \quad \mathcal{K}_F(e^{it}, e^{i\tau}) = f'(\tau) \mathcal{K}_h(f(t), f(\tau)).$$

To continue, observe first that

$$\frac{P(r, t)}{1-r} = \frac{1+r}{2\pi(1+r^2-2r\cos t)} \leq \frac{1}{\pi((1-r)^2+4r\sin^2 t/2)} \leq \frac{\pi}{4rt^2}$$

for $0 < r < 1$ and $t \in [-\pi, \pi]$.

On the other hand by (2.12), (2.14) and (2.23), for

$$\sigma = \pi |e^{if(t+\tau)} - e^{if(\tau)}|$$

and

$$|g|_\infty := \operatorname{ess\,sup}_t |g(t)|,$$

we obtain

$$|\mathcal{K}_F(e^{i(t+\tau)}, e^{i\tau})| \leq |h'|_\infty |f'|_\infty \int_0^\sigma \omega(x) dx.$$

Therefore

$$\begin{aligned}
(2.24) \quad \left| \mathcal{K}_F(e^{i(t+\tau)}, e^{i\tau}) \frac{P(r, t)}{1-r} \right| &\leq \frac{|h'|_\infty |f'|_\infty \pi}{4rt^2} \int_0^\sigma \omega(u) du \\
&\leq \frac{\sigma}{t} \frac{|h'|_\infty |f'|_\infty \pi}{4rt^2} \int_0^t \omega\left(\frac{\sigma}{t}x\right) dx \\
&\leq \frac{\pi |h'|_\infty |f'|_\infty^2}{4r} \frac{1}{t^2} \int_0^t \omega(\pi |f'|_\infty x) dx.
\end{aligned}$$

By (2.24), having in mind the equation (2.10), for $r > 1/2$ we obtain

$$\begin{aligned} \int_{-\pi}^{\pi} \left| \mathcal{K}_F(e^{it}, e^{i\tau}) \frac{P(r, \tau - t)}{1 - r} \right| dt &\leq \frac{2|h'|_{\infty}|f'|_{\infty}^2}{4r} \int_0^{\pi} \frac{1}{t^2} \int_0^t \omega(\pi|f'|_{\infty}u) du \\ &= \frac{|h'|_{\infty}|f'|_{\infty}^2}{2r} \int_0^{\pi} \left(\frac{\omega(\pi|f'|_{\infty}u)}{u} - \frac{\omega(\pi|f'|_{\infty}u)}{\pi} \right) du \\ &< M < \infty. \end{aligned}$$

According to Lebesgue Dominated Convergence Theorem, taking the limit under the integral sign in the last integral in (2.22) we obtain (2.17). \square

Lemma 2.7. *Let $u = P[F](z)$ be a Lipschitz continuous harmonic function between the unit disk \mathbf{U} and a $\mathcal{C}^{1,\mu}$ surface $\mathcal{M} \subset \mathbf{R}^n$ such that F is injective, and $\partial D = h(\mathbf{T}) \in \mathcal{C}^{1,\mu}$, where h is a $\mathcal{C}^{1,\mu}$ parametrization. Then for almost every $e^{i\varphi} \in \mathbf{T}$ we have*

$$(2.25) \quad \limsup_{r \rightarrow 1-0} |J_u(re^{i\varphi})| \leq C_h |f'(\varphi)| \int_{-\pi}^{\pi} \frac{|F(e^{i(\varphi+x)}) - F(e^{i\varphi})|^{1+\mu}}{\pi|e^{ix} - 1|^2} dx,$$

where $J_u(z)$ denotes the Jacobian of u at z , $F(e^{it}) = h(e^{if(t)})$, and

$$C_h = \frac{1}{(1 + \mu) \min_t |h'(t)|} \sup_{x \neq y} \frac{|h'(x) - h'(y)|}{|x - y|^{\mu}}.$$

Proof. It follows from Lemma 2.4 and Lemma 2.6. \square

3. THE MAIN RESULTS

Let $\gamma \subset \mathbf{R}^n$ be a closed rectifiable Jordan curve. Let $d_{\gamma}(a, b)$ be the length of the shorter Jordan arc of γ with endpoints $a, b \in \gamma$. We say that γ enjoys a λ -chord-arc condition for some constant $\lambda > 1$ (or γ is chord-arc) if for all $x, y \in \gamma$ there holds the inequality

$$(3.1) \quad d_{\gamma}(x, y) \leq \lambda |x - y|.$$

It is clear that if $\gamma \in \mathcal{C}^1$ then γ enjoys a chord-arc condition for some $\lambda_{\gamma} > 1$.

Lemma 3.1. *Assume that $\gamma \subset \mathbf{R}^n$ enjoys a chord-arc condition for some $\lambda > 1$ and that bounds a smooth Υ -isoperimetric surface \mathcal{M} ($\partial\mathcal{M} = \gamma$). Then for every K -q.c. normalized mapping u between the unit disk \mathbf{U} and \mathcal{M} there holds the inequality*

$$(3.2) \quad |u(z_1) - u(z_2)| \leq L_{\gamma}(K) |z_1 - z_2|^{\alpha}$$

for $z_1, z_2 \in \mathbf{T}$, $\alpha = \frac{8\Upsilon}{\pi K(1+2\lambda)^2}$ and

$$L_{\gamma}(K) = 4(1 + 2\lambda)2^{\alpha} \sqrt{\frac{2\pi K|\mathcal{M}|}{\log 2}},$$

where $|\mathcal{M}|$ is the area of the surface \mathcal{M} .

Proof. For $a \in \mathbf{C}$ and $r > 0$, put $D(a, r) := \{z : |z - a| < r\}$. It is clear that if $z_0 \in \mathbf{T} = \partial\mathbf{U}$, then, because of normalization, $u(\mathbf{T} \cap \overline{D(z_0, 1)})$ has common points with at most two of three arcs $\widehat{w_0 w_1}$, $\widehat{w_1 w_2}$ and $\widehat{w_2 w_0}$. (Here $w_0, w_1, w_2 \in \gamma$ divide γ into three arcs with the same length such that $u(1) = w_0$, $u(e^{2\pi i/3}) = w_1$, $u(e^{4\pi i/3}) = w_2$, and $\mathbf{T} \cap \overline{D(z_0, 1)}$ do not intersect at least one of three arcs defined by $1, e^{2\pi i/3}$ and $e^{4\pi i/3}$).

Let φ be a conformal mapping of a some neighborhood of \mathcal{M} onto the unit disk (this neighborhood exist according to the definition of disk-type surfaces with boundary). Then $u_1 = \varphi \circ u$ is a K quasiconformal mapping of the unit disk onto a domain with rectifiable boundary γ_1 which enjoys a chord-arc condition. According to [1, Theorem 5, p. 81], it admits a q.c. reflection. Let \tilde{u} be a q.c. mapping of the whole plane onto itself, such that $\tilde{u}|_{\mathbf{U}} = u_1$.

Let k_ρ denotes the arc of the circle $|z - z_0| = \rho < 1$ which lies in $|z| \leq 1$. Take $F(\rho, \varphi) = \tilde{u}_1(z_0 - \overline{z_0}\rho e^{i\varphi})$. Then for $n \in \mathbf{N}$ F is quasiconformal in $A_n = [\frac{1}{n}, n] \times [-\pi/2 - 1, \pi/2 + 1]$. Since $k_\rho \subset \{z_0 - \overline{z_0}\rho e^{i\varphi} : \varphi \in (-\pi/2, \pi/2)\}$, and F is absolutely continuous on almost every line $\{\rho\} \times [-\pi/2, \pi/2]$, it follows that u_1 is absolutely continuous on almost every k_ρ for $\rho > \frac{1}{n}$ as well as the mapping u . The conclusion is that u is absolutely continuous on almost every k_ρ for $\rho > 0$.

Let $l_\rho = |u(k_\rho)|$ denotes the length of $u(k_\rho)$. Let $\kappa_\rho = \{t \in [0, 2\pi] : z_0 + \rho e^{it} \in k_\rho\}$. By using polar coordinates and the Cauchy-Schwarz inequality, we have for almost every ρ

$$\begin{aligned} l_\rho^2 &= |u(k_\rho)|^2 = \left(\int_{k_\rho} |du| \right)^2 \\ &\leq \left(\int_{\kappa_\rho} |\nabla u(z_0 + \rho e^{i\varphi})| \rho d\varphi \right)^2 \\ &\leq \int_{\kappa_\rho} |\nabla u(z_0 + \rho e^{i\varphi})|^2 \rho d\varphi \cdot \int_{\kappa_\rho} \rho d\varphi. \end{aligned}$$

Let $\gamma_\rho := u(\mathbf{T} \cap D(z_0, \rho))$ and let $|\gamma_\rho|$ be its length. Assume w and w' are the endpoints of γ_ρ , i.e. of $u(k_\rho)$. Then $|\gamma_\rho| = d_\gamma(w, w')$ or $|\gamma_\rho| = |\gamma| - d_\gamma(w, w')$. If the first case holds, then since γ enjoys the λ -chord-arc condition, it follows $|\gamma_\rho| \leq \lambda|w - w'| \leq \lambda l_\rho$. Consider now the last case. Let $\gamma'_\rho = \gamma \setminus \gamma_\rho$. Then γ'_ρ contains one of the arcs $\widehat{w_0 w_1}$, $\widehat{w_1 w_2}$, $\widehat{w_2 w_0}$. Thus $|\gamma_\rho| \leq 2|\gamma'_\rho|$, and therefore

$$|\gamma_\rho| \leq 2\lambda l_\rho.$$

Since $l(k_\rho) \leq 2\rho\pi/2$, for $r \leq 1$, denoting $\Delta_r = \mathbf{U} \cap D(z_0, r)$, we have

$$\begin{aligned} \int_0^r \frac{l_\rho^2}{\rho} d\rho &\leq \pi \int_0^r \int_{\kappa_\rho} |\nabla u(z_0 + \rho e^{i\varphi})|^2 \rho d\varphi d\rho \\ (3.3) \quad &\leq \pi K \int_0^r \int_{\kappa_\rho} J_u(z_0 + \rho e^{i\varphi}) \rho d\varphi d\rho = \pi A(r)K, \end{aligned}$$

where $A(r)$ is the area of $u(\Delta_r)$. Using the first part of the proof, it follows that the length of boundary arc γ_r of $u(\Delta_r)$ does not exceed $2\lambda l_r$ which, according to the fact that $\partial u(\Delta_r) = \gamma_r \cup u(k_r)$, implies

$$(3.4) \quad |\partial u(\Delta_r)| \leq l_r + 2\lambda l_r.$$

Therefore, by the isoperimetric inequality

$$A(r) \leq \frac{|\partial u(\Delta_r)|^2}{4\Upsilon} \leq \frac{(l_r + 2\lambda l_r)^2}{4\Upsilon} = l_r^2 \frac{(1 + 2\lambda)^2}{4\Upsilon}.$$

Employing now (3.3) we obtain

$$F(r) := \int_0^r \frac{l_\rho^2}{\rho} d\rho \leq \pi K l_r^2 \frac{(1 + 2\lambda)^2}{4\Upsilon}.$$

Observe that for $0 < r \leq 1$ there holds $rF'(r) = l_r^2$. Thus

$$F(r) \leq \pi K r F'(r) \frac{(1 + 2\lambda)^2}{4\Upsilon}.$$

It follows that for

$$\alpha = \frac{8\Upsilon}{\pi K (1 + 2\lambda)^2}$$

there holds

$$\frac{d}{dr} \log(F(r) \cdot r^{-2\alpha}) \geq 0,$$

i.e. the function $F(r) \cdot r^{-2\alpha}$ is increasing. This yields

$$F(r) \leq F(1) r^{2\alpha} \leq \pi K |\mathcal{M}| r^{2\alpha}.$$

Now for every $r \leq 1$ there exists an $r_1 \in [r/\sqrt{2}, r]$ such that

$$F(r) = \int_0^r \frac{l_\rho^2}{\rho} d\rho \geq \int_{r/\sqrt{2}}^r \frac{l_\rho^2}{\rho} d\rho = l_{r_1}^2 \log \sqrt{2}.$$

Hence

$$l_{r_1}^2 \leq \frac{2\pi K |\mathcal{M}|}{\log 2} r^{2\alpha}.$$

If z is a point with $|z| \leq 1$ and $|z - z_0| = r/\sqrt{2}$, then by (3.4)

$$|u(z) - u(z_0)| \leq (1 + 2\lambda) l_{r_1}.$$

Therefore

$$|u(z) - u(z_0)| \leq H |z - z_0|^\alpha,$$

where

$$H = (1 + 2\lambda) 2^{\alpha/2} \sqrt{\frac{2\pi K |\mathcal{M}|}{\log 2}}.$$

Thus we have for $z_1, z_2 \in \mathbf{T}$ the inequality

$$(3.5) \quad |u(z_1) - u(z_2)| \leq 4H |z_1 - z_2|^\alpha.$$

□

We now prove the main result of this paper.

Theorem 3.2. *If $u : \mathbf{U} \rightarrow \mathcal{M}$ is a normalized K -quasiconformal harmonic mapping of the unit disk onto a $\mathcal{C}^{1,\mu}$ surface \mathcal{M} with $\mathcal{C}^{1,\mu}$ Jordan boundary γ , then u is Lipschitz and there exists a constant $L = L(K, \gamma)$ (satisfying inequality (3.17) below) such that*

$$|\nabla u| \leq L$$

and

$$|u(z) - u(w)| \leq KL|z - w|.$$

Proof. Let τ be a conformal mapping of the unit disk onto \mathcal{M} described in Remark 1.1. Then τ is bi-Lipschitz, i.e. there exists a constant $c_{\mathcal{M}} \geq 1$ such that

$$\frac{1}{c_{\mathcal{M}}} \leq \frac{|\tau(z) - \tau(w)|}{|z - w|} \leq c_{\mathcal{M}}, \quad z \neq w, z, w \in \mathbf{U}.$$

Let in addition φ_r be a conformal mapping of the unit disk onto $U_r = u^{-1}(\tau(r\mathbf{U}))$ such that the mapping $u_r = u \circ \varphi_r = P[F_r]$ is normalized. The mapping u_r is a K quasiconformal harmonic mapping of the unit disk onto the surface $\mathcal{M}_r \subset \mathcal{M}$ with boundary $\gamma_r = \partial\mathcal{M}_r$, satisfying λ' -chord-arc condition, where

$$(3.6) \quad \lambda' = \frac{\pi c_{\mathcal{M}}^2}{2}.$$

Namely let $a = \tau(re^{it})$ and $b = \tau(re^{is})$ be two points of the Jordan curve γ_r and let $d_{\gamma_r}(a, b)$ be the length of shorter Jordan arc of γ_r with endpoints a and b . Then

$$d_{\gamma_r}(a, b) \leq \int_{\widehat{[re^{it}, re^{is}]}} |\tau'(z)| |dz| \leq \frac{\pi c_{\mathcal{M}}}{2} |re^{is} - re^{it}| \leq \frac{\pi c_{\mathcal{M}}^2}{2} |a - b|.$$

Here $\widehat{[re^{it}, re^{is}]}$ is the shorter arc of $r\mathbf{T}$ with endpoints re^{it} and re^{is} .

Consider the parametrization $\tau_r : \mathbf{T} \rightarrow \gamma_r$ defined by $\tau_r(e^{it}) = \tau(re^{it})$, and write $\tau_r(t)$ instead of $\tau_r(e^{it})$. First of all

$$\tau_r'(t) = r\tau'(re^{it})$$

and there exists a constant c_{τ} such that

$$(3.7) \quad \frac{r}{c_{\tau}} \leq |\tau_r'(t)| \leq rc_{\tau}.$$

On the other hand,

$$C_{\tau_r} := \sup_{t \neq s} \frac{|\tau_r'(s) - \tau_r'(t)|}{|t - s|^{\mu}} = \sup_{t \neq s} \frac{r|\tau'(re^{is}) - \tau'(re^{it})|}{|t - s|^{\mu}} \leq rC_{\mathcal{M}},$$

where $C_{\mathcal{M}}$ is defined in (1.5).

Let

$$L_r = |F_r'|_{\infty} := \max\{|F_r'(e^{is})| : s \in [0, 2\pi]\} = |F_r'(e^{it})|.$$

First of all

$$(3.8) \quad |F_r(e^{i(t+x)}) - F_r(e^{it})| \leq \frac{\pi}{2} L_r |e^{ix} - 1|.$$

By (2.1), (2.25) (taking $h := \tau_r$) and (3.7) we obtain

$$|F'_r(t)|^2 \leq \frac{C_{\tau_r} K}{2rc_\tau} |F'_r(t)| \int_{-\pi}^{\pi} \frac{|F_r(e^{i(t+x)}) - F_r(e^{it})|^{1+\mu}}{|e^{ix} - 1|^2} dx,$$

i.e.

$$|F'_r(t)| \leq \frac{C_{\tau_r} K}{2rc_\tau} \int_{-\pi}^{\pi} \frac{|F_r(e^{i(t+x)}) - F_r(e^{it})|^{1+\mu}}{|e^{ix} - 1|^2} dx.$$

Hence for

$$(3.9) \quad C_r = \frac{C_{\tau_r} K}{2rc_\tau},$$

and for β satisfying $0 < \beta < 1$, in view of (3.8), we obtain

$$(3.10) \quad \begin{aligned} L_r &\leq C_r \int_{-\pi}^{\pi} \frac{|F_r(e^{i(t+x)}) - F_r(e^{it})|^{1+\mu}}{|e^{ix} - 1|^{1-\mu}} \frac{dx}{|e^{ix} - 1|^{1+\mu}} \\ &\leq C_r \int_{-\pi}^{\pi} \frac{|F_r(e^{i(t+x)}) - F_r(e^{it})|^{1+\mu-\beta}}{|e^{ix} - 1|^{1+\mu-\beta}} L^\beta \frac{dx}{|e^{ix} - 1|^{1-\mu}}. \end{aligned}$$

Thus

$$L_r/L_r^\beta \leq C_r \int_{-\pi}^{\pi} \frac{|F_r(e^{i(t+x)}) - F_r(e^{it})|^{1+\mu-\beta}}{|e^{ix} - 1|^{1+\mu-\beta}} \frac{dx}{|e^{ix} - 1|^{1-\mu}}.$$

In view of Definition 2.5, harmonic surfaces are 1-isoperimetric. By Lemma 3.1 F_r is Hölder continuous with exponent

$$(3.11) \quad \alpha' = \frac{2\Upsilon}{\pi K(1+2\lambda')^2} \geq \frac{2}{\pi K(1+2\lambda')^2}.$$

Choose β , $0 < \beta < 1$, sufficiently close to 1, so that

$$\sigma = (\alpha' - 1)(1 + \mu - \beta) + \mu - 1 > -1.$$

For example,

$$\beta = 1 - \frac{\mu\alpha'}{2 - \alpha'},$$

and consequently,

$$\sigma = \frac{\mu\alpha'}{2 - \alpha'} - 1.$$

Because γ_r is a λ' chord-arc curve, we get

$$L_r^{1-\beta} \leq C_r \cdot (L_{\gamma_r}(K))^{1+\mu-\beta} \int_{-\pi}^{\pi} |e^{ix} - 1|^\sigma dx = C'_r,$$

and hence

$$(3.12) \quad L_r \leq (C'_r)^{1/(1-\beta)} = (C'_r)^{\frac{2-\alpha'}{\mu\alpha'}}.$$

On the other hand

$$\begin{aligned}
C'_r &\leq C_r(L_\gamma(K))^{1+\mu-\beta} \frac{2^{1+\sigma}\pi^{3/2}\sec(\pi\sigma/2)}{\Gamma(1/2-s/2)\Gamma(1+s/2)} \\
&\leq \frac{C_{\mathcal{M}}K}{2c_\tau}(L_\gamma(K))^{1+\mu-\beta} \frac{2^{1+\sigma}\pi^{3/2}\sec(\pi\sigma/2)}{\Gamma(1/2-\sigma/2)\Gamma(1+\sigma/2)} \\
&\leq \frac{C_{\mathcal{M}}K}{c_\tau} \frac{2^\sigma\pi}{1+\sigma}(L_\gamma(K))^{1+\mu-\beta}.
\end{aligned}$$

Thus, in view of (3.6) and (3.11) we obtain

$$(3.13) \quad \sup_r L_r < \infty.$$

Since F_r is smooth we have

$$\frac{\partial u_r}{\partial t}(\rho e^{it}) = P[F'_r](\rho e^{it}).$$

From (2.2) and (2.7) it follows

$$(3.14) \quad |\nabla u_r(e^{it})| \leq K \left| \frac{\partial u_r}{\partial t}(e^{it}) \right| \leq K|F'_r|_\infty.$$

To continue, observe that ∇u_r is harmonic and bounded and there holds

$$\nabla u_r(z) = P[\nabla u_r(e^{it})](z).$$

Thus for $|h| = 1$

$$|\nabla u_r(z)h| = |P[\nabla u_r(e^{it})h](z)| \leq P[|\nabla u_r(e^{it})h|](z).$$

It follows that

$$(3.15) \quad |\nabla u_r| \leq K|F'_r|_\infty = KL_r.$$

Combining (3.15) and (3.13) we obtain

$$\sup\{|\nabla u(z)| : |z| < 1\} < \infty.$$

By using the fact proved in the previous proof that u is Lipschitz continuous, and proceeding again the previous proof: setting L instead of L_r , f instead of f_r , and an arc length parametrization g of γ instead of τ_r , and using the fact that a smooth curve is chord-arc curve for some λ , we obtain that

$$\text{ess sup}\{|F'(t)| : 0 \leq t \leq 2\pi\} = L(K, \lambda, \Upsilon) =: L,$$

and

$$(3.16) \quad |\nabla u| \leq K|F'|_\infty = KL.$$

See remark below for an explicit estimate of L . □

Remark 3.3. Since $4|\mathcal{M}| \leq |\gamma|^2$, the previous proof yields the following estimate of a Lipschitz constant L for a normalized K -quasiconformal harmonic

mapping between the unit disk and a disk-type surface \mathcal{M} bounded by a Jordan curve $\gamma \in \mathcal{C}^{1,\mu}$ satisfying a λ -chord-arc condition.

$$(3.17) \quad L \leq 8 \left(KC_\gamma \frac{\pi(2-\alpha)}{2\mu\alpha} \right)^{\frac{2-\alpha}{\mu\alpha}} \left\{ 4(1+2\lambda)|\gamma| \sqrt{\frac{\pi K}{\log 4}} \right\}^{\frac{2}{\alpha}},$$

where

$$\alpha = \frac{8\Upsilon}{\pi K(1+2\lambda)^2}, \quad \Upsilon \geq 1$$

and

$$C_\gamma = \max_{s \neq t} \frac{|g'(t) - g'(s)|}{|t - s|^\mu}.$$

Here g is an arc length parametrization of γ . See [39], [38], [25] and [17] for more explicit (more precise) constants, in the special case where γ is the unit circle. We can express λ in terms of g as follows

$$(3.18) \quad \lambda = \sup_{s \neq t} \frac{|s - t|}{|g(s) - g(t)|}.$$

It is clear that $\lambda < \infty$, because

$$\lim_{(s,t) \rightarrow (\tau,\tau)} \frac{|s - t|}{|g(s) - g(t)|} = \frac{1}{|g'(\tau)|} = 1.$$

The following corollary of Theorem 3.2 gives a quantitative estimate of the lipschitz constant of a conformal mapping, parameterizing a minimal surface with smooth boundary (see [34] and [28] for existential proof of this fact and [44] and [45] for related results).

Corollary 3.4. *Let $u(x, y) = (u^1, \dots, u^n) : \mathbf{U} \rightarrow \mathcal{M}$ be normalized isotherm coordinates of a minimal surface \mathcal{M} spanning a Jordan curve $\gamma \in \mathcal{C}^{1,\mu}$ and assume that g is an arc-length parametrization of γ . Then*

$$|u_x| = |u_y| \leq L,$$

where

$$L = 8 \left\{ \frac{C_\gamma(-3/4 + \lambda(1+\lambda))\pi}{2\mu} \right\}^{\frac{-3/4 + \lambda(1+\lambda)}{\mu}} \left\{ \frac{4(1+2\lambda)|\gamma|}{\sqrt{\log 4}} \right\}^{(1/2+\lambda)^2},$$

and λ is defined in (3.18). In particular if $\mu = 1$, we obtain

$$L = 8 \left\{ \frac{\kappa_\gamma \pi}{2} \left[-\frac{3}{4} + \lambda(1+\lambda) \right] \right\}^{-\frac{3}{4} + \lambda(1+\lambda)} \left\{ \frac{4(1+2\lambda)|\gamma|}{\sqrt{\log 4}} \right\}^{(1/2+\lambda)^2},$$

where κ_γ is the largest curvature of γ defined as

$$\kappa_\gamma = \operatorname{ess\,sup}_{0 \leq s \leq |\gamma|} |g''(s)|.$$

Proof. Note first that, a minimal surface is not necessarily regular. Thus we cannot apply directly Theorem 3.2. By [34, Theorem 1] (see also [29, Theorem 1]), u is Lipschitz continuous. Therefore the proof of Theorem 3.2 gives desired estimates. The constant is better than that of (3.17), because for minimal surfaces we have $\Upsilon = \pi$, $K = 1$ and $4\pi|\mathcal{M}| \leq |\gamma|^2$. \square

By using Theorem 3.2 and isotherm coordinates near the points in the boundary we obtain

Theorem 3.5. *If $w : \mathcal{N} \rightarrow \mathcal{M}$ is a K -quasiconformal harmonic mapping between $\mathcal{C}^{1,\mu}$ surfaces \mathcal{N} and \mathcal{M} with $\mathcal{C}^{1,\mu}$ and compact boundaries, then w is Lipschitz.*

Proof. Under the conditions of theorem, w has a continuous extension to the boundary. Take $p \in \partial\mathcal{N}$, let $q = w(p) \in \partial\mathcal{M}$ and choose a disk-type surface M_0 with smooth boundary, containing an open Jordan arc $\gamma_q \subset \partial\mathcal{M}$, such that $q \in \gamma_q$. Let φ be a conformal mapping of the unit disk onto $N_0 = u^{-1}(\mathcal{M}_0)$, such that $\varphi(1) = p$. Then $u = w \circ \varphi$ is a K -quasiconformal harmonic mapping of the unit disk onto $\mathcal{M}_0 \subset \mathcal{M}$. By Theorem 3.2 u is Lipschitz. According to Remark 1.1 φ is bi-Lipschitz in some small neighborhood of 1. This implies that w is Lipschitz in some neighborhood of p . As $\partial\mathcal{N}$ is compact, it follows that w is Lipschitz near the boundary $\partial\mathcal{N}$ of \mathcal{N} . The conclusion is that, w is Lipschitz. \square

3.1. Remarks. Our main theorem has an important assumption that the surface \mathcal{M} is a $\mathcal{C}^{1,\mu}$ regular surface. Since the Lipschitz constant depends only on the boundary of \mathcal{M} , it would be interesting to check whether the assumption that \mathcal{M} is regular is essential. In the case that \mathcal{M} is a minimizing surface spanning a sufficiently smooth Jordan curve, then \mathcal{M} is a regular surface, i.e. \mathcal{M} has not branch points inside neither on the boundary (see for example [46] and reference therein for this topic). We expect that, our main result still hold assuming only that the corresponding mapping $u = P[F]$ is a q.c. harmonic mapping such that $F(\mathbf{T})$ is a $\mathcal{C}^{1,\mu}$ Jordan curve. On the other hand, harmonic surfaces can have branch points as well as q.c. harmonic surfaces because minimal surfaces that are not minimizing surfaces are not free of branch points. Therefore, probably, the corresponding harmonic q.c. mapping is not bi-Lipschitz, without the assumption that the surface is regular. In a recent result of the author ([22]), it is proved that, if the surface is at least $\mathcal{C}^{2,\mu}$ regular, then such mapping is bi-Lipschitz, and we expect that the class $\mathcal{C}^{2,\mu}$ can be replaced by $\mathcal{C}^{1,\mu}$.

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